

LAGRANGIAN L -STABILITY OF LAGRANGIAN TRANSLATING SOLITONS

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ABSTRACT. In this paper, we prove that any Lagrangian translating soliton is Lagrangian L -stable.

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1. INTRODUCTION

Recent years, motivated by the problem of existence of special Lagrangian submanifolds, Lagrangian mean curvature flow has attracted much attention. It was proved by Chen-Li ([2]) and Wang ([13]) that there is no finite time Type I singularity for almost calibrated Lagrangian mean curvature flow. Therefore, there are many works concentrating on Type II singularities of Lagrangian mean curvature flow, especially, on Lagrangian translating solitons ([5], [6], [9], [11], [12], etc.).

An n -dimensional submanifold Σ^n in \mathbb{R}^{n+k} is called a *translating soliton* if there exists a constant vector $\mathbf{T} \in \mathbb{R}^{n+k}$, such that

$$(1.1) \quad \mathbf{T}^\perp = \mathbf{H}$$

holds on Σ , where \mathbf{H} is the mean curvature vector of Σ^n in \mathbb{R}^{n+k} .

Similar to that of self-shrinkers ([4]), one can also study the translating solitons from variational viewpoint. Actually, translating solitons can be viewed as critical points of the following functional:

$$(1.2) \quad F(\Sigma) = \int_{\Sigma} e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu,$$

where \mathbf{x} is the position vector in \mathbb{R}^{n+k} and $d\mu$ is the induced area element on Σ . Then it is natural to define stability of translating solitons. Shahriyari ([10]) proved that any translating graph in \mathbb{R}^3 is L -stable.

A translating soliton Σ^n in \mathbb{C}^n is called a *Lagrangian translating soliton* if it is also a Lagrangian submanifold of \mathbb{C}^n . In [14], L. Yang proved that any Lagrangian translating soliton is Hamiltonian L -stable. In this paper, we prove that it is in fact Lagrangian L -stable:

Theorem 1.1. *Any Lagrangian translating soliton is Lagrangian L -stable.*

Key words and phrases. Lagrangian translating soliton, Lagrangian L -stable.

The proof of Theorem 1.1 relies crucially on that the variation is Lagrangian. There are many examples for Lagrangian translating solitons ([3], [7], etc.). By Theorem 1.1, they are all Lagrangian L -stable. One natural question is whether we can find examples which are in fact L -stable (not just only Lagrangian L -stable). In [1], we showed that the Grim Reaper cylinder $\Gamma \times \mathbb{R}^{n-1}$ is L -stable in \mathbb{R}^{n+1} , where Γ is the Grim Reaper in the plane. This is actually true for any mean convex translating soliton Σ^n in \mathbb{R}^{n+1} . In this paper, we will show that:

Theorem 1.2. *The Lagrangian Grim Reaper cylinder $\Gamma \times \mathbb{R}$ in \mathbb{C}^2 is L -stable.*

For the relations between Lagrangian F -stable self-shrinkers and Hamiltonian F -stable self-shrinkers, we refer to [8].

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2. PRELIMINARIES

In this section, we will recall some results for the first variation and second variation formulas. Since the proofs can be found in Section 4 of [1] with $f = \langle \mathbf{T}, \mathbf{x} \rangle$, where we dealt with more general cases (see also [14]), we omit the details here.

Recall that the F -functional is defined by

$$F(\Sigma) = \int_{\Sigma} e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu.$$

The following first variation formula is known (Proposition 4.1 of [1]):

Proposition 2.1. *Let $\Sigma_s^n \subset \mathbb{R}^{n+k}$ be a smooth compactly supported variation of Σ with normal variational vector field \mathbf{V} , then*

$$(2.1) \quad \frac{d}{ds}\bigg|_{s=0} F(\Sigma_s) = \int_{\Sigma} \langle \mathbf{T}^{\perp} - \mathbf{H}, \mathbf{V} \rangle e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu.$$

In particular, Σ is a critical point of F if and only if $\mathbf{T}^{\perp} = \mathbf{H}$, i.e., Σ is a translating soliton in \mathbb{R}^{n+k} .

For the second variation formula, we have (see (4.17) of [1]):

Theorem 2.2. *Suppose that Σ is a critical point of F . If $\Sigma_s^n \subset \mathbb{R}^{n+k}$ be a smooth compactly supported variation of Σ with normal variational vector field \mathbf{V} , then the*

second variation formula is given by

$$(2.2) \quad F'' := \frac{d^2}{ds^2}|_{s=0} F(\Sigma_s) = - \int_{\Sigma} \langle L\mathbf{V}, \mathbf{V} \rangle e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu.$$

Here, the stability operator L is defined on a normal vector field \mathbf{V} on M by

$$(2.3) \quad L\mathbf{V} = \left(\Delta V^\alpha + \langle \mathbf{T}, \nabla V^\alpha \rangle + g^{ik} g^{jl} h_{ij}^\alpha h_{kl}^\beta V^\beta \right) e_\alpha,$$

where $\{e_\alpha\}$ is a local orthonormal frame of the normal bundle $N\Sigma$, g_{ij} is the induced metric on Σ and $\mathbf{V} = V^\alpha e_\alpha$.

Definition 2.1. A translating soliton Σ^n in \mathbb{R}^{n+k} is said to be **L -stable** if for every compactly supported normal variational vector field \mathbf{V} , we have

$$F'' = - \int_{\Sigma} \langle L\mathbf{V}, \mathbf{V} \rangle e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \geq 0.$$

Now we turn to Lagrangian translating solitons. Let $\bar{\omega}$ and J be the standard Kähler form and complex structure on \mathbb{C}^n , respectively. A submanifold Σ^n is said to be a *Lagrangian* submanifold of \mathbb{C}^n , if $\bar{\omega}|_{\Sigma} = 0$, or equivalently, J maps the tangent space of Σ on to its normal space at each point of Σ . For a Lagrangian submanifold, there is a canonical correspondence between the sections of the normal bundle and the space of 1-forms on Σ :

$$\begin{aligned} \Gamma(N\Sigma) &\longrightarrow \Lambda^1(\Sigma) \\ \mathbf{V} &\longleftrightarrow \theta_{\mathbf{V}} := -i_{\mathbf{V}}\bar{\omega}. \end{aligned}$$

A normal vector field \mathbf{V} is a *Lagrangian variation* if $\theta_{\mathbf{V}}$ is closed; a normal vector field \mathbf{V} is a *Hamiltonian variation* if $\theta_{\mathbf{V}}$ is exact.

Definition 2.2. A Lagrangian translating soliton Σ^n in \mathbb{C}^n is said to be **Lagrangian (resp. Hamiltonian) L -stable** if for every compactly supported normal Lagrangian (resp. Hamiltonian) variation \mathbf{V} , we have

$$F'' = - \int_{\Sigma} \langle L\mathbf{V}, \mathbf{V} \rangle e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \geq 0.$$

3. LAGRANGIAN L -STABILITY OF LAGRANGIAN TRANSLATING SOLITONS

In this section, we will prove Theorem 1.1. First, we would like to rewrite the second variation formula for Lagrangian variations.

Let $(\mathbb{C}^n, \bar{g}, J, \bar{\omega})$ be the complex Euclidean space with standard metric \bar{g} , complex structure J and Kähler form $\bar{\omega}$ such that $\bar{g} = \bar{\omega}(\cdot, J\cdot)$. Given any Lagrangian submanifold Σ^n in \mathbb{C}^n , we choose a local orthonormal frame $\{e_i\}_{i=1}^n$ of $T\Sigma$, and set $\nu_i = Je_i$. Then $\{\nu_i\}_{i=1}^n$ forms a local orthonormal frame of the normal bundle $N\Sigma$. The frame

can be chosen so that at a fixed point $x \in \Sigma$, we have $\nabla_{e_i} e_j = 0$, where ∇ is the induced connection on Σ . The second fundamental form is defined by

$$h_{ijk} = \bar{g}(\bar{\nabla}_{e_i} e_j, \nu_k),$$

which is symmetric in i, j and k . The mean curvature vector is given by

$$\mathbf{H} = H_k \nu_k = h_{iik} \nu_k.$$

Let $\{\omega^i\}_{i=1}^n$ be the dual basis of $\{e_i\}_{i=1}^n$. Then for any normal vector field $\mathbf{V} = V_i \nu_i$, we have the correspondence

$$\theta_{\mathbf{V}} := -i_{\mathbf{V}} \bar{\omega} = V_i \omega^i.$$

Since $d\theta_{\mathbf{V}} = \nabla_{e_i} V_j \omega^j \wedge \omega^i$, we see that

Proposition 3.1. *A normal vector field \mathbf{V} of a Lagrangian submanifold Σ^n in \mathbb{C}^n is a Lagrangian variation if and only if $\nabla_{e_i} V_j = \nabla_{e_j} V_i$.*

Using the above notations, we see that the stability operator (2.3) can be rewritten as

$$(3.1) \quad L\mathbf{V} = (\Delta V_i + \langle \mathbf{T}, \nabla V_i \rangle + h_{kli} h_{klj} V_j) \nu_i.$$

Therefore, we have

Proposition 3.2. *A Lagrangian translating soliton Σ^n in \mathbb{C}^n is Lagrangian L -stable if and only if for every compactly supported normal Lagrangian variation $\mathbf{V} = V_i \nu_i$, we have*

$$(3.2) \quad F'' = - \int_{\Sigma} (V_i \Delta V_i + V_i \langle \mathbf{T}, \nabla V_i \rangle + h_{kli} h_{klj} V_i V_j) e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \geq 0.$$

Now, we can prove the first main result. For the purpose of convenience, we rewrite it here:

Theorem 3.3. *Any Lagrangian translating soliton is Lagrangian L -stable.*

Proof: By Proposition 3.2, it suffices to prove that (3.2) holds for every compactly supported Lagrangian variation $\mathbf{V} = V_i \nu_i$. Since $\mathbf{V} = V_i \nu_i$ is Lagrangian, by Proposition 3.1, we see that $\nabla_{e_i} V_j = \nabla_{e_j} V_i$. By Ricci identity, we have

$$(3.3) \quad \Delta V_i = \nabla_j \nabla_j V_i = \nabla_j \nabla_i V_j = \nabla_i \nabla_j V_j + R_{jijk} V_k = \nabla_i \nabla_j V_j + R_{ik} V_k,$$

where R_{ik} is the Ricci curvature of the induced metric on Σ . By Gauss equation, we have that

$$R_{ijkl} = h_{pik} h_{pjl} - h_{pil} h_{pjk},$$

which implies that

$$(3.4) \quad R_{ik} = g^{jl} R_{ijkl} = H_p h_{pik} - h_{pji} h_{pjk}.$$

Putting (3.4) into (3.3) yields

$$\Delta V_i = \nabla_i \nabla_j V_j + R_{ik} V_k = \nabla_i \nabla_j V_j + H_p h_{pik} V_k - h_{pji} h_{pjk} V_k.$$

Therefore, we have

$$(3.5) \quad F'' = - \int_{\Sigma} (V_i \nabla_i \nabla_j V_j + V_i \langle \mathbf{T}, e_j \rangle \nabla_j V_i + H_p h_{pij} V_i V_j) e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu.$$

Integrating by part, we can compute the first term on the right hand side of (3.5) as:

$$(3.6) \quad \begin{aligned} & - \int_{\Sigma} V_i \nabla_i \nabla_j V_j e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \\ &= \int_{\Sigma} (\nabla_i V_i \nabla_j V_j + V_i \nabla_j V_j \nabla_i \langle \mathbf{T}, \mathbf{x} \rangle) e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \\ &= \int_{\Sigma} \left[\left(\sum_{j=1}^n \nabla_j V_j \right)^2 + \left(\sum_{j=1}^n \nabla_j V_j \right) \left(\sum_{i=1}^n \langle \mathbf{T}, e_i \rangle V_i \right) \right] e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu. \end{aligned}$$

On the other hand, from the translating soliton equation (1.1), we can easily see that $H_p = \langle \mathbf{T}, \nu_p \rangle$. Therefore, using the fact that $\nabla_{e_i} V_j = \nabla_{e_j} V_i$, the second term on the right hand side of (3.5) can be computed as:

$$(3.7) \quad \begin{aligned} & - \int_{\Sigma} V_i \langle \mathbf{T}, e_j \rangle \nabla_j V_i e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu = - \int_{\Sigma} (\nabla_i V_j) V_i \langle \mathbf{T}, e_j \rangle e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \\ &= \int_{\Sigma} [(\nabla_i V_i) \langle \mathbf{T}, e_j \rangle V_j + V_i V_j \nabla_i \langle \mathbf{T}, e_j \rangle + V_i V_j \langle \mathbf{T}, e_j \rangle \nabla_i \langle \mathbf{T}, \mathbf{x} \rangle] e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \\ &= \int_{\Sigma} [(\nabla_i V_i) \langle \mathbf{T}, e_j \rangle V_j + V_i V_j \langle \mathbf{T}, h_{pij} \nu_p \rangle + V_i V_j \langle \mathbf{T}, e_j \rangle \langle \mathbf{T}, e_i \rangle] e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \\ &= \int_{\Sigma} \left[\left(\sum_{j=1}^n \nabla_j V_j \right) \left(\sum_{i=1}^n \langle \mathbf{T}, e_i \rangle V_i \right) + H_p h_{pij} V_i V_j + \left(\sum_{i=1}^n \langle \mathbf{T}, e_i \rangle V_i \right)^2 \right] e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu. \end{aligned}$$

Here, we used the fact that $\bar{\nabla}_{e_i} e_j = h_{pij} \nu_j$ at a fixed point by the choice of the frame. Putting (3.6) and (3.7) into (3.5) yields

$$F'' = \int_{\Sigma} \left(\sum_{j=1}^n \nabla_j V_j + \sum_{i=1}^n \langle \mathbf{T}, e_i \rangle V_i \right)^2 e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \geq 0.$$

This finishes the proof of the theorem.

Q.E.D.

4. THE LAGRANGIAN GRIM REAPER CYLINDER

In the previous section, we proved that any Lagrangian translating soliton is Lagrangian L -stable. However, it is not clear that whether they are L -stable. In this section, as an example, we will show that the Grim Reaper cylinder $\Gamma \times \mathbb{R}$ is in fact L -stable in \mathbb{C}^2 .

First recall that the Grim Reaper Γ in the plane is defined by

$$\gamma : \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \longrightarrow \mathbb{C}$$

$$x \longrightarrow \gamma(x) = (-\log \cos x, x).$$

Then the Grim Reaper cylinder $\Gamma \times \mathbb{R}$ is defined by

$$\begin{aligned} \Phi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} &\longrightarrow \mathbb{C}^2 \\ (x, y) &\longrightarrow \Phi(x, y) = (-\log \cos x, x, y, 0). \end{aligned}$$

We will see that it is a Lagrangian translating soliton and is L -stable.

Theorem 4.1. *The Grim Reaper cylinder $\Sigma = \Gamma \times \mathbb{R}$ is a Lagrangian translating soliton of \mathbb{C}^2 and is L -stable.*

Proof: By the definition of Φ , the tangent space of Σ is spanned by

$$\Phi_x = (\tan x, 1, 0, 0), \quad \Phi_y = (0, 0, 1, 0).$$

The orthonormal basis of the normal space can be taken as

$$e_3 = (\cos x, -\sin x, 0, 0), \quad e_4 = (0, 0, 0, -1).$$

The induced metric can be represented as

$$(4.1) \quad (g_{ij})_{1 \leq i, j \leq 2} = \begin{pmatrix} \frac{1}{\cos^2 x} & 0 \\ 0 & 1 \end{pmatrix}, \quad (g^{ij})_{1 \leq i, j \leq 2} = \begin{pmatrix} \cos^2 x & 0 \\ 0 & 1 \end{pmatrix}.$$

The induced area form is given by

$$(4.2) \quad d\mu = \sqrt{\det(g_{ij})} dx dy = \frac{1}{\cos x} dx dy.$$

Since

$$\Phi_{xx} = \left(\frac{1}{\cos^2 x}, 0, 0, 0\right), \quad \Phi_{xy} = (0, 0, 0, 0), \quad \Phi_{yy} = (0, 0, 0, 0),$$

from $h_{ij}^\alpha = \langle \Phi_{ij}, e_\alpha \rangle$, we see that the second fundamental form are given by

$$(4.3) \quad h_{xx}^3 = \frac{1}{\cos x}, \quad h_{xx}^3 = h_{xy}^3 = h_{yy}^3 = h_{xx}^4 = h_{xy}^4 = h_{yy}^4 = 0.$$

Therefore,

$$H^3 = g^{ij} h_{ij}^3 = g^{xx} h_{xx}^3 = \cos x, \quad H^4 = g^{ij} h_{ij}^4 = 0,$$

and the mean curvature vector is given by

$$\mathbf{H} = H^3 e_3 + H^4 e_4 = \cos x e_3.$$

Now if we take $\mathbf{T} = (1, 0, 0, 0) \in \mathbb{C}^2$, then

$$\mathbf{T}^\perp = \langle \mathbf{T}, e_3 \rangle e_3 + \langle \mathbf{T}, e_4 \rangle e_4 = \cos x e_3 = \mathbf{H}.$$

Therefore, Σ is a translating soliton in \mathbb{C}^2 .

Recall that the standard complex structure in \mathbb{C}^2 is given by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Since

$$J\Phi_x = (1, -\tan x, 0, 0) = \frac{1}{\cos x} e_3, \quad J\Phi_y = (0, 0, 0, -1) = e_4,$$

we see that Σ is a Lagrangian translating soliton in \mathbb{C}^2 .

Next we will show that Σ is L -stable. Since

$$\Delta v + \langle \mathbf{T}, \nabla v \rangle = e^{-\langle \mathbf{T}, \mathbf{x} \rangle} \operatorname{div}_\Sigma (e^{\langle \mathbf{T}, \mathbf{x} \rangle} \nabla v),$$

for any smooth function v on Σ , we can see easily from Theorem 2.2 that Σ is L -stable if and only if

$$(4.4) \quad \int_\Sigma g^{ik} g^{jl} h_{ij}^\alpha h_{kl}^\beta V^\alpha V^\beta e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu \leq \int_\Sigma \sum_\alpha |\nabla V^\alpha|^2 e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu$$

holds for every compactly supported normal variation vector field $\mathbf{V} = V^\alpha e_\alpha$.

In our case, $\langle \mathbf{T}, \mathbf{x} \rangle = \langle (1, 0, 0, 0), (-\log \cos x, x, y, 0) \rangle = -\log \cos x$ so that

$$(4.5) \quad e^{\langle \mathbf{T}, \mathbf{x} \rangle} = \frac{1}{\cos x}.$$

By (4.1) and (4.3), we have

$$g^{ik} g^{jl} h_{ij}^\alpha h_{kl}^\beta V^\alpha V^\beta = g^{xx} g^{xx} h_{xx}^3 h_{xx}^3 V^3 V^3 = \cos^2 x (V^3)^2.$$

Combining with (4.2) and (4.5), we get that

$$(4.6) \quad \int_\Sigma g^{ik} g^{jl} h_{ij}^\alpha h_{kl}^\beta V^\alpha V^\beta e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu = \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (V^3)^2 dx dy.$$

On the other hand, using (4.1), we compute

$$|\nabla V^\alpha|^2 = g^{ij} \frac{\partial}{\partial x^i} V^\alpha \frac{\partial}{\partial x^j} V^\alpha = \cos^2 x \left(\frac{\partial}{\partial x} V^\alpha \right)^2 + \left(\frac{\partial}{\partial y} V^\alpha \right)^2.$$

Therefore,

$$(4.7) \quad \begin{aligned} \int_\Sigma \sum_\alpha |\nabla V^\alpha|^2 e^{\langle \mathbf{T}, \mathbf{x} \rangle} d\mu &= \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(\frac{\partial}{\partial x} V^3 \right)^2 + \left(\frac{\partial}{\partial x} V^4 \right)^2 \right] dx dy \\ &\quad + \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cos^2 x} \left[\left(\frac{\partial}{\partial y} V^3 \right)^2 + \left(\frac{\partial}{\partial y} V^4 \right)^2 \right] dx dy \end{aligned}$$

Note that $V^3, V^4 \in C_0^\infty((-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R})$. In particular, for each fixed y , we have $V^3(\cdot, y) \in C_0^\infty((-\frac{\pi}{2}, \frac{\pi}{2}))$. By Wirtinger inequality, we have for each $y \in \mathbb{R}$ that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (V^3(x, y))^2 dx \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial}{\partial x} V^3(x, y) \right)^2 dx$$

Integrating with respect to y yields

$$(4.8) \quad \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (V^3(x, y))^2 dx dy \leq \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\partial}{\partial x} V^3(x, y) \right)^2 dx dy.$$

Combining (4.6), (4.7) and (4.8), we see that (4.4) holds for every compactly supported normal variation vector field $\mathbf{V} = V^3 e_3 + V^4 e_4$. This shows that the Lagrangian translating soliton Σ is L -stable. Q.E.D.

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